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Extremum seeking with bounded update rates[☆]

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ABSTRACT

In this work, we present a form of extremum seeking (ES) in which the unknown function being minimized enters the system's dynamics as the argument of a cosine or sine term, thereby guaranteeing known bounds on update rates and control efforts. We present general n-dimensional optimization and stabilization results as well as 2D vehicle control, with bounded velocity and control efforts. For application to autonomous vehicles, tracking a source in a GPS denied environment with unknown orientation, this ES approach allows for smooth heading angle actuation, with constant velocity, and in application to a unicycle-type vehicle results in control ability as if the vehicle is fully actuated. Our stability analysis is made possible by the classic results of Kurzweil, Jarnik, Sussmann, and Liu, regarding systems with highly oscillatory terms. In our stability analysis, we combine the averaging results with a semiglobal practical stability result under small parametric perturbations developed by Moreau and Aeyels.

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1. Introduction

Motivation. Extremum seeking (ES), as a real-time non-model-based optimization approach, has seen growth following significant theoretical advances during the past decade, starting with the proof of local convergence by Krstić and Wang [1] and extension to semiglobal convergence by Tan, Nešić, and Mareels [2]. ES has now been used in diverse applications with unknown/uncertain systems [3,4], such as active flow control [5,6], aeropropulsion [7], cooling systems [8,9], wind energy [10], photovoltaics [11], electromagnetic valve actuation [12], human exercise machines [13], controlling Tokamak plasmas [14], PID gain tuning [15], enhancing mixing in magnetohydrodynamic channel flows [16], beam matching [17], laser pulse shaping [18] and high voltage converter modulator output voltage optimization [19].

Recent work by Dürr et al. [20] combined the Lie bracket-based averaging results of Gurvits and Li [21] with results of Moreau and Aeyels [22], providing a technique for Lyapunov function based ES analysis. In [23], the application of ES was then expanded beyond minimization, to stabilization of unknown, open-loop unstable systems, solving the problem of model-independent semiglobal exponential practical stabilization for time-varying and nonlinear systems. In [24], by introducing a non-smooth ES scheme, persistent oscillations, which plague all ES schemes, were made to decay as a system approaches equilibrium.

Because ES is designed to perform with unknown systems, one of the most promising applications is for the control of autonomous vehicles, and has been demonstrated as a powerful tool for steering vehicles towards a source in GPS-denied environments [25–27].

Despite the mentioned theoretical advancements and applications, one limitation which remains in all ES schemes is the uncertainty of convergence rate and control effort. This is due to the fact that an unknown function, whether it is the unknown output of a system which is being minimized, or a Lyapunov candidate for a system which is being stabilized, enters the control scheme in an affine way.

Results of the paper. In this work we present a new ES scheme, in which the uncertainty is confined to the argument of a sine/cosine function, resulting in guaranteed bounds on update rate in minimum seeking and control effort in stabilization. In order to prove our stability conditions we introduce the results of Kurzweil and Jarnik [28], and Sussmann and Liu [29,30].

The controller that we develop, in the case of minimization of a measurable, but unknown output function $J(\theta)$ of a dynamic system, is given by

$$\dot{\theta}_i = u_i = \sqrt{\alpha_i \omega_i} \cos\left(\omega_i t + k_i J\right). \tag{1}$$

In this scheme, a high frequency (ω_i) dither is applied to parameter θ_i, whose magnitude is proportional to (after averaging) α_i, k_i can be thought of as the controller gain. These parameters are discussed in more detail below.

In the case of stabilization of a system of the form

$$\dot{x} = f(x,t) + g(x,t)u, \tag{2}$$

the controller's components are chosen as

$$u_i = \sqrt{\alpha_i \omega_i} \cos\left(\omega_i t + k_i V(x)\right), \tag{3}$$

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where V is a Lyapunov function candidate. In these two cases, the closed loop systems, on average, satisfy the dynamics:

$$\dot{\bar{\theta}}_i = -\frac{k\alpha}{2}\frac{\partial J}{\partial \bar{\theta}_i}, \tag{4}$$

$$\dot{\bar{x}} = f(\bar{x}, t) - \frac{k\alpha}{2} g(\bar{x}, t) g^T(\bar{x}, t)\frac{\partial V(\bar{x})}{\partial \bar{x}}. \tag{5}$$

Note that both the update rate (1) and control effort (3) have bounds of the form $\sqrt{\alpha\omega}$, independent of $J(\theta)$ or $V(x)$.

Next, we consider the particular case of 2D vehicle control, in which an unknown, but measurable function $J(x, y)$, whose value depends on vehicle position (x, y) is to be minimized or maximized, in a GPS denied environment. The controller that we develop towards this goal is given by:

$$\dot{x} = \sqrt{\alpha\omega}\cos\left(\omega t + kJ(x, y) + \theta_0\right) \tag{6}$$

$$\dot{y} = \sqrt{\alpha\omega}\sin\left(\omega t + kJ(x, y) + \theta_0\right) \tag{7}$$

where θ_0 is an arbitrary initial vehicle orientation. The resulting closed loop system, on average, has dynamics

$$[\dot{x}, \dot{y}]^T = -\frac{k\alpha}{2}\left(\nabla J\right)^T, \tag{8}$$

and performs gradient descent towards a local minimum of $J(x, y)$. Note that in this case, the vehicle velocity $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\alpha\omega}$, is constant, and the vehicle performs smooth, unicycle-type motion in circular trajectories.

Intuitively, the mechanism behind the convergence of system (6), (7) is easy to see. At any time, t, the system's velocity vector is given by $\sqrt{\alpha\omega}\left[\cos(\theta(t)), \sin(\theta(t))\right]^T$, where $\theta(t) = \omega t + kJ(x, y) + \theta_0$. In this case, the rate of change of the heading, $\theta(t)$ is given by:

$$\dot{\theta} = \omega + k\frac{\partial J}{\partial t}. \tag{9}$$

A large value of ω keeps the system spinning around, looking in all directions. When the system's trajectory is heading in a direction such that J is decreasing, then $\frac{\partial J}{\partial t} < 0$, and the rate of change of $\theta(t)$ is decreased. On the other hand, when the system's trajectory is heading in a direction such that J is increasing, then $\frac{\partial J}{\partial t} > 0$, and the rate of change of $\theta(t)$ is increased. By this mechanism, on average, the system spends more time pointing in the right direction, and quickly turning away from the wrong direction. The overall result is convergence towards the minimum value. The system is made to approach a maximum by replacing k with $-k$.

In our 2D vehicle analysis, we focus on applying the simple algorithm as stated above, and therefore rely on direct heading angle actuation. It is possible to use the same approach to instead regulate the heading angle rate by adding additional layers of filtering to the scheme above. The new scheme is more complicated and stability/convergence analysis requires a singular perturbation version of the averaging theory presented here or standard averaging combined with singular perturbation techniques that lead to local results. Although we do not pursue a lengthy combined singular perturbation and averaging analysis in this paper, we do present the algorithm design for heading angle rate actuation and demonstrate its ability with a simulation.

Organization. In Section 2 we provide background on the averaging results of Kurzweil, Jarnik, Sussmann and Liu. We also review stability results of Moreau and Aeyels. In Section 3 we present our optimization results. In Section 4 we provide nonlinear system stabilization results. In Section 5 we present our vehicle control scheme. Finally, in Section 6 we present several examples of a vehicle tracking both a stationary and a mobile source in an unknown environment with external disturbances.

2. Background on averaging and stability

In this section we begin by recalling the main theorems which are necessary for calculating the averaged dynamics of the closed loop systems with highly oscillatory terms, for which standard averaging techniques are not applicable. Theorem 1 is a general result, of which the special case given by Corollary 1 is applied throughout the paper. The results of Moreau and Aeyels which follow are necessary in order to provide a relationship between the stability of the averaged and actual system dynamics.

2.1. Averaging results of Kurzweil, Jarnik, Sussmann, and Liu

Theorem 1 ([28–30]). *For $T \in (0, \infty)$, and a compact set $K \subset \mathbb{R}^n$, consider the differential equation:*

$$\dot{x} = f(x, t) + \sum_{i=1}^{n} f_i(x, t)\varphi_{i,k}(t), \qquad x(0) = x_0, \tag{10}$$

where the functions

$$f(x, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n \tag{11}$$

$$f_i(x, t) : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n \tag{12}$$

$$\varphi_{i,k}(t) : [0, T] \to \mathbb{R} \tag{13}$$

are continuous and Lipschitz, and Df_i, $D^2 f_i$, $\frac{\partial f_i}{\partial t}$, $\frac{\partial}{\partial t}Df_i$, $\frac{\partial}{\partial t}D^2 f_i$, are continuous and bounded, where $D = \frac{\partial}{\partial x}$.

If the functions $\varphi(t)$ are continuous and their integrals satisfy:

$$\Phi_{i,k}(t) = \int_0^t \varphi_{i,k}(\tau)d\tau \to 0 \quad \text{uniformly as } k \to \infty, \tag{14}$$

and there exist measurable functions $\lambda_{i,j}(t) : [0, T] \to \mathbb{R}$ such that for all $s, t \in [0, T]$

$$\lim_{k\to\infty}\int_0^t \varphi_{j,k}(\tau)\Phi_{i,k}(\tau)d\tau = \int_0^t \lambda_{i,j}(\tau)d\tau, \quad \text{uniformly}, \tag{15}$$

$$\left|\Phi_{i,k}(t) - \Phi_{i,k}(s)\right| \le L_2|t - s|^{\eta}, \quad 0 < \eta < \frac{1}{2}, \tag{16}$$

$$\left|\int_0^t \varphi_{j,k}(\tau)\left[\Phi_{i,k}(\tau) - \Phi_{i,k}(s)\right]d\tau\right| \le L_1|t - s|^{\gamma}, \tag{17}$$

where $1-\eta < \gamma \le 1$. Then, for all $t \in [0, T]$ and $x \in K$, the sequence of solutions of (10):

$$x_k(t) = x_0 + \int_0^t \left(f(x_k, \tau) + \sum_{i=1}^{n} f_i(x_k, \tau)\varphi_{i,k}(\tau)\right)d\tau \tag{18}$$

converges uniformly with respect to k, over $(x, t) \in K \times [0, T]$ to the solution $x(t)$ satisfying:

$$\dot{x} = f(x, t) - \sum_{i,j=1}^{n}\lambda_{i,j}(t)\left(Df_i(x, t)\right)f_j(x, t), \qquad x(0) = x_0. \tag{19}$$

The following special case of Theorem 1, with sine and cosine highly oscillatory terms, is used throughout the paper.

Corollary 1. *For $T \in [0, \infty)$, and any compact set $K \subset \mathbb{R}^n$ such that the functions $f(x, t)$, $f_i(x, t)$ satisfy the assumptions of Theorem 1, for any $\nu, \delta > 0$, there exists M such that for all $k > M$, the trajectory $x(t)$ of the system*

$$\dot{x} = f(x, t) + \sum_{i=1}^{n} f_i(x, t)(kk_i)^{\nu}\cos\left((kk_i)^{2\nu}t\right) - \sum_{i=1}^{n} g_i(x, t)(kk_i)^{\nu}\sin\left((kk_i)^{2\nu}t\right), \tag{20}$$

and the trajectory $\bar{x}(t)$ of the system

$$\dot{\bar{x}} = f(\bar{x}, t) - \frac{1}{2}\sum_{i \neq j}^{n}\left[f_i(\bar{x}, t), g_j(\bar{x}, t)\right], \qquad \bar{x}(0) = x(0) \in K, \quad (21)$$

satisfy the convergent trajectories property:

$$\max_{t \in [0,T]} |x(t) - \bar{x}(t)| < \delta, \quad (22)$$

where $k \in \mathbb{N}$, $k_i \in \mathbb{R}$ such that $\hat{k}_i \neq \hat{k}_j$, and

$$\left[f_i(\bar{x}, t), g_j(\bar{x}, t)\right] = \frac{\partial g_j}{\partial \bar{x}} f_i - \frac{\partial f_i}{\partial \bar{x}} g_j, \quad (23)$$

is the Lie bracket of the functions f_i and g_i.

Proof. All of the conditions of Theorem 1 are satisfied with $\eta = \frac{1}{2}$, $\gamma = 1$,

$$\varphi_{i,k} = (kk_i)^{\nu} \cos\left((kk_i)^{2\nu} t\right), \quad (24)$$

$$\hat{\varphi}_{i,k} = -(kk_i)^{\nu} \sin\left((kk_i)^{2\nu} t\right), \quad (25)$$

$$\Phi_{i,k}(t) = \frac{1}{(kk_i)^{\nu}} \sin((kk_i)^{2\nu} t), \quad (26)$$

$$\hat{\Phi}_{i,k}(t) = \frac{1}{(kk_i)^{\nu}} \cos((kk_i)^{2\nu} t), \quad (27)$$

with

$$\lambda_{i,j} = \begin{cases} \frac{1}{2} : \text{mixed terms } \varphi_{i,k}\hat{\Phi}_{j,k},\ \hat{\varphi}_{i,k}\Phi_{j,k} \text{ s.t. } i = j \\ 0 : \text{mixed terms } \varphi_{i,k}\hat{\Phi}_{j,k},\ \hat{\varphi}_{i,k}\Phi_{j,k} \text{ s.t. } i \neq j \\ 0 : \text{all non-mixed terms } \varphi_{i,k}\Phi_{j,k},\ \hat{\varphi}_{i,k}\hat{\Phi}_{j,k}. \end{cases} \quad (28)$$

□

2.2. Stability results of Moreau and Aeyels

We recall the following definitions as in Moreau and Aeyels [22]. In what follows, given a system

$$\dot{x} = f(t, x), \quad (29)$$

$\psi(t, t_0, x_0)$ denotes the solution of (29) which passes through the point x_0 at time t_0. In conjunction with (29), we consider systems of the form

$$\dot{x} = f^{\epsilon}(t, x) \quad (30)$$

whose trajectories are denoted as $\phi^{\epsilon}(t, t_0, x_0)$.

Definition 1 (*Converging Trajectories Property*)**.** The systems (29) and (30) are said to satisfy the converging trajectories property if for every $\hat{T} \in (0, \infty)$ and compact set $K \subset \mathbb{R}^n$ satisfying $\{(t, t_0, x_0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : t \in [t_0, t_0 + \hat{T}], x_0 \in K\} \subset \text{Dom}\psi$, for every $d \in (0, \infty)$ there exists $\epsilon^{\star}$ such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in K$ and for all $\epsilon \in (0, \epsilon^{\star})$,

$$\|\phi^{\epsilon}(t, t_0, x_0) - \psi(t, t_0, x_0)\| < d, \quad \forall t \in [t_0, t_0 + \hat{T}]. \quad (31)$$

In an approach similar to Moreau and Aeyels, we define two forms of stability for system (30):

Definition 2 (*ϵ-Semiglobal Practical Uniform Ultimate Boundedness with Ultimate Bound δ ((ϵ, δ)-SPUUB)*)**.** The origin of (30) is said to be (ϵ, δ)-SPUUB if there exists $\delta > 0$ such that the following three conditions are satisfied:

- (ϵ, δ)-*Uniform Stability:* For every $c_2 \in (\delta, \infty)$ there exists $c_1 \in (0, \infty)$ and $\hat{\epsilon} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\epsilon \in (0, \hat{\epsilon})$,

  $\|\psi^{\epsilon}(t, t_0, x_0)\| < c_2 \quad \forall t \in [t_0, \infty).$

- (ϵ, δ)-*Uniform Ultimate Boundedness:* For every $c_1 \in (0, \infty)$ there exists $c_2 \in (\delta, \infty)$ and $\hat{\epsilon} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\epsilon \in (0, \hat{\epsilon})$,

  $\|\psi^{\epsilon}(t, t_0, x_0)\| < c_2 \quad \forall t \in [t_0, \infty).$

- (ϵ, δ)-*Global Uniform Attractivity:* For all $c_1, c_2 \in (\delta, \infty)$ there exists $T \in (0, \infty)$ and $\hat{\epsilon} \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\epsilon \in (0, \hat{\epsilon})$,

  $\|\psi^{\epsilon}(t, t_0, x_0)\| < c_2 \quad \forall t \in [t_0 + T, \infty).$

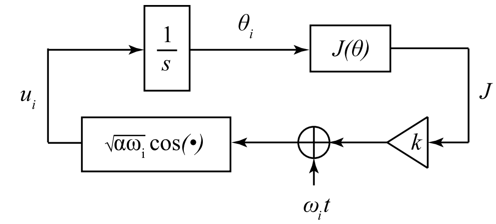

Fig. 1. ES scheme for the ith component θ_i of θ.

Definition 3 (*ϵ-Semiglobal Practical Uniform Asymptotic Stability (ϵ-SPUAS)*)**.** The origin of (30) is said to be ϵ-SPUAS if it is (ϵ, δ)-SPUUB for all $\delta > 0$, in which case all of the conditions of Definition 2 are replaced with a lower bound of 0 on c_2, instead of δ.

With these definitions the following result of Moreau and Aeyels [22] is used in the analysis that follows.

Theorem 2 (*[22]*)**.** *If systems (30) and (29) satisfy the converging trajectories property and if the origin is a GUAS equilibrium point of (29), then the origin of (30) is ϵ-SPUAS.*

Corollary 2. *If the origin of system (19) is GUAS, then the origin of system (10) is $\frac{1}{k}$-SPUAS.*

Proof. By Theorem 1 the solutions of (10) and (19) satisfy the converging trajectories property for any $T \in [0, \infty)$. Since the origin of (19) is GUAS, by Theorem 2, the origin of (10) is $\frac{1}{k}$-SPUAS. □

Corollary 3 (*[22]*)**.** *If systems (10) and (19) satisfy the converging trajectories property and if the origin is a δ-GUUB point of (19), then the origin of (10) is $\left(\frac{1}{k}, \delta\right)$-SPUUB.*

Proof. By Theorem 1 the solutions of (10) and (19) satisfy the converging trajectories property for any $T \in [0, \infty)$. The rest of the proof is a slight modification of the proof found in Moreau and Aeyels [22], with a lower bound of δ instead of 0 on the choices of c_2, b_2 and c_3, with details available from the author. □

3. Extremum seeking for unknown map

Consider the problem of locating an extremum point of the function $J(\theta) : \mathbb{R}^n \to \mathbb{R}$, for $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. We assume that $J(\theta)$ has a global extremum such that there exists a unique $\theta^{\star}$ for which:

$$\nabla J|_{\theta^{\star}} = 0 \quad \text{and} \quad \nabla J \neq 0,\ \forall \theta \neq \theta^{\star}. \quad (32)$$

Theorem 3. *Consider the ES scheme shown in Fig. 1 (for maximum seeking we replace k_i with $-k_i$):*

$$\dot{\theta}_i = \sqrt{\alpha_i \omega_i} \cos\left(\omega_i t + k_i J(\theta)\right), \quad (33)$$

where $\omega_i = \omega\hat{\omega}_i$ such that $\hat{\omega}_i \neq \hat{\omega}_j\ \forall i \neq j$ and J satisfies (32). The point $\theta^{\star}$ is $\frac{1}{\omega}$-SPUAS.

Proof. By expanding

$$\cos\left(\omega_i t + k_i J\right) = \cos(\omega_i t)\cos(k_i J) - \sin(\omega_i t)\sin(k_i J) \quad (34)$$

we rewrite the θ_i dynamics as

$$\dot{\theta}_i = \sqrt{\omega_i}\cos(\omega_i t)\sqrt{\alpha_i}\cos(k_i J) - \sqrt{\omega_i}\sin(\omega_i t)\sqrt{\alpha_i}\sin(k_i J), \quad (35)$$

and applying Corollary 1 (with respect to ω and with $\nu = 0.5$) the trajectory of system (33) uniformly converges to the trajectory of

$$\dot{\bar{\theta}}_i = -\frac{k_i\alpha_i}{2}\frac{\partial J(\bar{\theta})}{\partial\bar{\theta}_i}, \quad (36)$$

where we have used the fact that mismatched terms of the form $\cos(\omega_i t)\sin(\omega_j t)$, $\forall i \neq j$, and terms of the form $\cos(\omega_i t)\cos(\omega_j t)$, and $\sin(\omega_i t)\sin(\omega_j t)$, $\forall i, j$ have averaged to zero. Combining all the θ_i components we then get:

$$\dot{\bar{\theta}} = -\frac{k\alpha}{2}(\nabla J)^T, \quad (37)$$

where $k\alpha$ is the diagonal matrix with entries $k_i\alpha_i$. □

4. Nonlinear MIMO systems with matched uncertainties

In this section we study multi-input systems with the same number of controls and states. We use this class to illustrate clearly how to deal with nonlinearities that are not only unknown but also have arbitrary growth (super-linear, exponential, or even faster than exponential).

Theorem 4. *Consider the following system over a compact set $K \subset \mathbb{R}^n$:*

$$\dot{x} = f(x,t) + G(x,t)u(x,t), \quad (38)$$

where $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$, and $u(x,t), f(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$, $G(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ and let there exist $\zeta \in \mathcal{K}$, and $\eta \in \mathcal{K}_\infty$ such that $f(x,t)$ and $G(x,t)$ satisfy the following bounds for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$:

$$G(x,t)G^T(x,t) \geq \zeta(|x|)I, \quad \forall x \in K \quad (39)$$

$$\sup_{x\in K}|f(x,t)| \leq \eta(|x|). \quad (40)$$

If k and α are chosen such that

$$k\alpha > \sup_{x\in K}\frac{1}{\zeta(|x|)}, \quad (41)$$

then the vector-valued controller with components

$$u_i = \sqrt{\alpha\omega_i}\cos(\omega_i t + kV(x)), \quad (42)$$

where $\omega_i = \omega\hat{\omega}_i$ such that $\hat{\omega}_i \neq \hat{\omega}_j\ \forall i \neq j$, and

$$V(x) = \int_0^{|x|}\eta(r)dr, \quad (43)$$

renders the origin of (38), (42) $\left(\frac{1}{\omega}, \zeta^{-1}\left(\frac{1}{k\alpha}\right)\right)$-SPUUB.

Remark 1. The following proof is based on an application of Theorem 1, which is an existence result regarding a large enough value of ω for our desired result to hold. Clearly, from the form of (38), (42), in order for stabilization to be possible, we must choose ω large enough such that $\zeta(|x|)\sqrt{\alpha\omega} > |f|$. Although this detail is glossed over in our existence result, exactly such a requirement can be found if one writes out the proof of Theorem 1 for this particular system, in which, after integration by parts, terms of the form $\frac{|f|}{\sqrt{\omega}}$ will appear, which approach zero as $\omega \to \infty$.

Proof. We expand (42) as

$$u_i = \sqrt{\alpha\omega_i}\cos(\omega_i t)\cos(kV(x)) - \sqrt{\alpha\omega_i}\sin(\omega_i t)\sin(kV(x)), \quad (44)$$

and apply Corollary 1. By an analysis similar to that in the proof of [23, Theorem 5], we evaluate the averaged system as:

$$\dot{\bar{x}} = f(\bar{x},t) - \frac{k\alpha}{2}G(\bar{x},t)G^T(\bar{x},t)\eta(|\bar{x}|)\frac{\bar{x}}{|\bar{x}|}, \quad (45)$$

where we have used the fact that

$$\frac{\partial V(\bar{x})}{\partial\bar{x}} = \eta(|\bar{x}|)\frac{\bar{x}^T}{|\bar{x}|}. \quad (46)$$

With the Lyapunov function candidate

$$W(\bar{x}) = \frac{|\bar{x}|^2}{2}, \quad (47)$$

we get

$$\dot{W}(\bar{x}) = \bar{x}^T\dot{\bar{x}} = \bar{x}^T f - k\alpha\frac{\eta(|\bar{x}|)}{|\bar{x}|}\bar{x}^T GG^T\bar{x}. \quad (48)$$

From (40) we have

$$\left|\bar{x}^T f\right| \leq |\bar{x}|\,|f| \leq |\bar{x}|\eta(|\bar{x}|) \quad (49)$$

and from (39) we have that

$$k\alpha\frac{\eta(|\bar{x}|)}{|\bar{x}|}\bar{x}^T G(\bar{x},t)G^T(\bar{x},t)\bar{x} \geq k\alpha\frac{\eta(|\bar{x}|)}{|\bar{x}|}\beta(|\bar{x}|)\,|\bar{x}|^2. \quad (50)$$

Plugging (49) and (50) into the equation for $\dot{W}(\bar{x})$ we get

$$\begin{aligned}\dot{W}(\bar{x}) &\leq |\bar{x}|\eta(|\bar{x}|) - k\alpha\beta(|\bar{x}|)\,|\bar{x}|\eta(|\bar{x}|)\\ &= (1 - k\alpha\beta(|\bar{x}|))|\bar{x}|\eta(|\bar{x}|),\end{aligned} \quad (51)$$

therefore by our choice of $k\alpha$ as in (41), we guarantee that (51) is negative definite outside of a ball or radius $\beta^{-1}\left(\frac{1}{k\alpha}\right)$ about the origin. Therefore, the averaged system (45) is ultimately bounded with ultimate bound $\beta^{-1}\left(\frac{1}{k\alpha}\right)$. By Corollary 3, system (38) is $\left(\frac{1}{\omega}, \beta^{-1}\left(\frac{1}{k\alpha}\right)\right)$-SPUUB. □

5. 2D vehicle control

In this section we consider a vehicle in a GPS-denied environment, unaware of its own orientation, whose goal is to reach the location of the minimum of $J(x,y)$, where $J(x,y)$ is a detectable value, whose analytic form is unknown.

Theorem 5. *If the function $J(x,y)$ has a global minimum at $(x^\star, y^\star)$, such that*

$$\nabla J|_{(x^\star,y^\star)} = 0, \quad \nabla J \neq 0,\ \forall(x,y) \neq (x^\star,y^\star), \quad (52)$$

then for any $\delta > 0$, by a sufficiently large choice of $k\alpha$ the point $(x^\star, y^\star)$ is $\left(\frac{1}{\omega}, \delta\right)$-SPUUB relative to the system $(x(t), y(t))$, as shown in Fig. 2:

$$\dot{x} = \sqrt{\alpha\omega}\cos(\omega t + kJ(x,y) + \theta_0) \quad (53)$$

$$\dot{y} = \sqrt{\alpha\omega}\sin(\omega t + kJ(x,y) + \theta_0) \quad (54)$$

where θ_0 is an unknown initial orientation.

Remark 2. In the analysis that follows, it becomes apparent that the value of the arbitrary initial orientation, θ_0, is irrelevant, when we make the simplification:

$$\sin^2(kJ + \theta_0) + \cos^2(kJ + \theta_0) = 1,$$

therefore for notational convenience, and without loss of generality, from now on we set $\theta_0 = 0$.

Proof. We expand

$$\cos(\omega t + kJ) = \cos(\omega t)\cos(kJ) - \sin(\omega t)\sin(kJ) \quad (55)$$

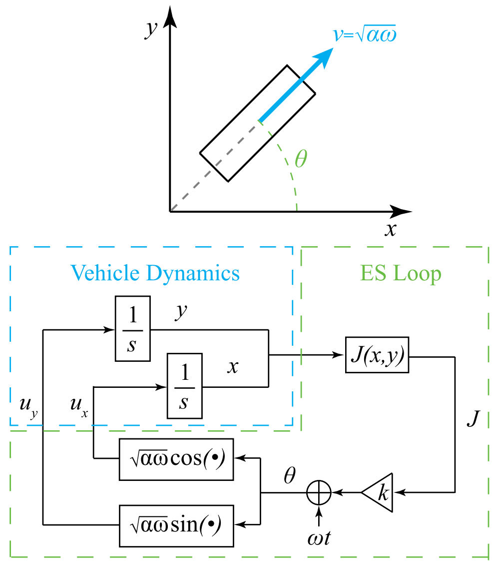

Fig. 2. Velocity actuated ES control scheme.

and

$$\sin(\omega t + kJ) = \cos(\omega t)\sin(kJ) + \sin(\omega t)\cos(kJ) \tag{56}$$

and rewrite (53), (54) as

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \sqrt{\omega}\cos(\omega t)\begin{bmatrix}\sqrt{\alpha}\cos(kJ)\\ \sqrt{\alpha}\sin(kJ)\end{bmatrix} + \sqrt{\omega}\sin(\omega t)\begin{bmatrix}-\sqrt{\alpha}\sin(kJ)\\ \sqrt{\alpha}\cos(kJ)\end{bmatrix}. \tag{57}$$

By Theorem 1, the trajectory of (57) uniformly converges to the trajectory of

$$\begin{bmatrix}\dot{\bar{x}}\\ \dot{\bar{y}}\end{bmatrix} = \frac{\alpha}{2}D\left(\begin{bmatrix}-\sin(kJ)\\ \cos(kJ)\end{bmatrix}\right)\begin{bmatrix}\cos(kJ)\\ \sin(kJ)\end{bmatrix} - \frac{\alpha}{2}D\left(\begin{bmatrix}\cos(kJ)\\ \sin(kJ)\end{bmatrix}\right)\begin{bmatrix}-\sin(kJ)\\ \cos(kJ)\end{bmatrix}. \tag{58}$$

We expand the right side of (58) as

$$\frac{k\alpha}{2}\begin{bmatrix}-\frac{\partial J}{\partial \bar{x}}\cos(kJ) & -\frac{\partial J}{\partial \bar{y}}\cos(kJ)\\ -\frac{\partial J}{\partial \bar{x}}\sin(kJ) & -\frac{\partial J}{\partial \bar{y}}\sin(kJ)\end{bmatrix}\begin{bmatrix}\cos(kJ)\\ \sin(kJ)\end{bmatrix} - \frac{k\alpha}{2}\begin{bmatrix}-\frac{\partial J}{\partial \bar{x}}\sin(kJ) & -\frac{\partial J}{\partial \bar{y}}\sin(kJ)\\ \frac{\partial J}{\partial \bar{x}}\cos(kJ) & \frac{\partial J}{\partial \bar{y}}\cos(kJ)\end{bmatrix}\begin{bmatrix}-\sin(kJ)\\ \cos(kJ)\end{bmatrix}, \tag{59}$$

which simplifies to

$$\dot{\bar{x}} = -\frac{k\alpha}{2}\left(\frac{\partial J}{\partial \bar{x}}\cos^2(kJ) + \frac{\partial J}{\partial \bar{x}}\sin^2(kJ)\right), \tag{60}$$

$$\dot{\bar{y}} = -\frac{k\alpha}{2}\left(\frac{\partial J}{\partial \bar{y}}\sin^2(kJ) + \frac{\partial J}{\partial \bar{y}}\cos^2(kJ)\right). \tag{61}$$

Applying the identity

$\cos^2(\cdot) + \sin^2(\cdot) = 1$,

we arrive at the average system dynamics

$$\begin{bmatrix}\dot{\bar{x}}\\ \dot{\bar{y}}\end{bmatrix} = -\frac{k\alpha}{2}\left(\nabla J(\bar{x},\bar{y})\right)^T. \tag{62}$$

Therefore, by Theorem 1 the trajectory $(x(t), y(t))$ of system (53)–(54) uniformly converges to the trajectory $(\bar{x}(t), \bar{y}(t))$, of the system

$$\dot{\bar{x}} = -\frac{k\alpha}{2}\frac{\partial J}{\partial \bar{x}}, \qquad \bar{x}(0) = x(0), \tag{63}$$

$$\dot{\bar{y}} = -\frac{k\alpha}{2}\frac{\partial J}{\partial \bar{y}}, \qquad \bar{y}(0) = y(0), \tag{64}$$

and therefore, for any $\delta > 0$, by choosing arbitrarily large values of $k\alpha$ we may ultimately bound $(\bar{x}, \bar{y})$ within a δ neighborhood of $(x^\star, y^\star)$. □

Remark 3. Although the results presented above are for functions having a stationary extremum, they are easily extended to systems where the extremum point varies with time, such as the case of trajectory tracking, in which the cost is the distance between a mobile agent and its target.

Corollary 4. *Consider a function* $f(x, y, t) = \left(f_x(x, y, t), f_y(x, y, t)\right)^T$*, over a compact set* $(x, y) \in K \subset \mathbb{R}^2$*, which is continuous with respect to t and Lipschitz continuous with respect to* (x, y)*. If the function* $J(x, y, t)$ *has a global minimum at* $(x^\star(t), y^\star(t)) \in K\ \forall t$*, such that the location of the minimum point has bounded velocity* $|\dot{x}^\star|, |\dot{y}^\star| < M$*, and*

$$\nabla J|_{(x^\star(t), y^\star(t))} = 0, \tag{65}$$

$$\nabla J \neq 0, \quad \forall (x(t), y(t)) \neq (x^\star(t), y^\star(t)), \tag{66}$$

then for any $\delta > 0$*, by a sufficiently large choice of* $k\alpha$*,* $(x^\star(t), y^\star(t))$ *is* $\left(\frac{1}{\omega}, \delta\right)$*-SPUUB relative to the system:*

$$\dot{x} = f_x(x, y, t) + \sqrt{\alpha\omega}\cos\left(\omega t + kJ(x, y, t)\right) \tag{67}$$

$$\dot{y} = f_y(x, y, t) + \sqrt{\alpha\omega}\sin\left(\omega t + kJ(x, y, t)\right) \tag{68}$$

where θ_0 *is an unknown initial orientation.*

Proof. We define the error variables $e_x(t) = x(t) - x^\star(t)$ and $e_y(t) = y(t) - y^\star(t)$ and show, by the same proof as above, that the trajectory of the error system of (67)–(68) uniformly converges to the trajectory of

$$\dot{\bar{e}}_x = f_x(\bar{e}_x + x^\star, \bar{e}_y + y^\star, t) - \frac{k\alpha}{2}\frac{\partial J}{\partial \bar{e}_x} + \dot{x}^\star(t), \tag{69}$$

$$\dot{\bar{e}}_y = f_y(\bar{e}_x + x^\star, \bar{e}_y + y^\star, t) - \frac{k\alpha}{2}\frac{\partial J}{\partial \bar{e}_y} + \dot{y}^\star(t). \tag{70}$$

Because the velocities $|\dot{x}^\star|$ and $|\dot{y}^\star|$ are bounded, and the function $f(x, y, t)$ is bounded on the compact set K, for any $\delta > 0$, by choosing arbitrarily large values of $k\alpha$ we may ultimately bound $(\bar{x}, \bar{y})$ within a δ neighborhood of $(x^\star, y^\star)$. □

6. 2D vehicle simulations

6.1. Stationary source seeking

In order to illustrate the behavior of the control system for a vehicle with unknown orientation we first demonstrate the scheme in an environment without external disturbance, in which the goal is to seek the stationary minimum of an unknown, but measurable function. We consider the system

$$\dot{x} = \sqrt{\alpha\omega}\cos\left(\omega t + kJ(x, y)\right), \qquad x(0) = 1 \tag{71}$$

$$\dot{y} = \sqrt{\alpha\omega}\sin\left(\omega t + kJ(x, y)\right), \qquad y(0) = -1, \tag{72}$$

where $J = x^2 + y^2$, $\alpha = \frac{1}{2}$, $k = 2$, $\theta(0) = 1.2$, and $\omega = 25$.

The simulation results are shown in Fig. 3. By showing the system's trajectory (x, y), alongside that of the averaged system, $(\bar{x}, \bar{y})$, it is easy to see that the convergence is along a gradient descent towards the minimum of $J(x, y)$.

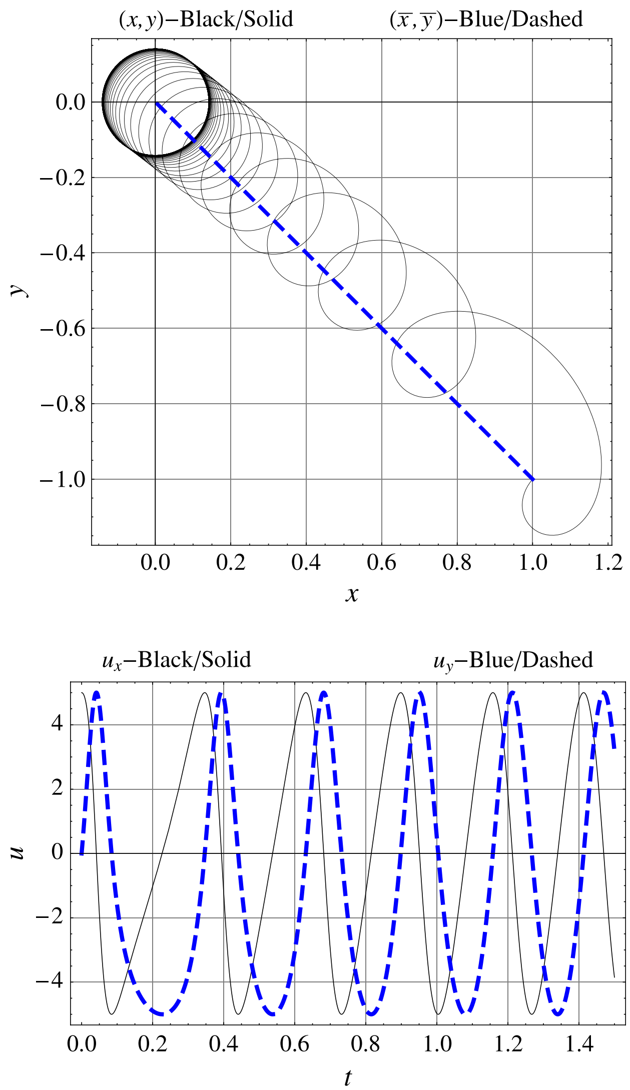

Fig. 3. Tracking of a stationary source is shown for 10 s, along with control effort for the first 1.5 s. The initial trajectory of $(x(t), y(t))$ is far from circular, as the system rotates slower while heading in the correct direction (towards decreasing J), which is the mechanism of convergence. This is also clearly seen in the control effort, where both the sine and cosine terms are initially asymmetric, changing faster or slower, depending on the heading direction.

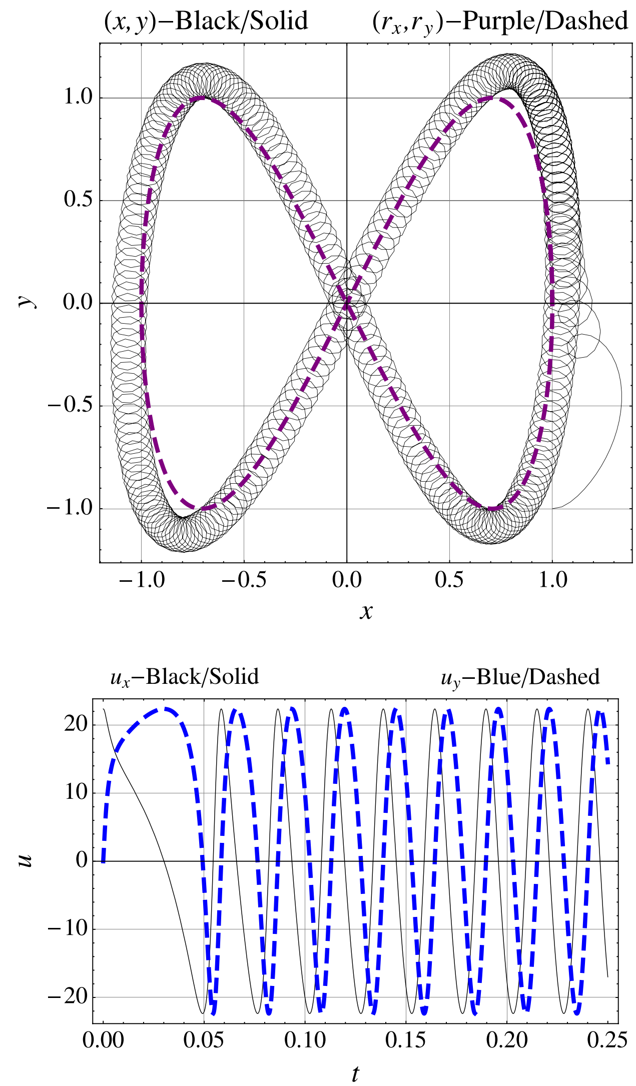

Fig. 4. Tracking of a moving source, despite external disturbances, is shown for 7 s, along with control effort for the first 0.25 s. The initial trajectory of $(x(t), y(t))$ is far from circular, the system rotates slower while heading in the correct direction (towards decreasing J), as it makes a large arc towards the location of the minimum of J. This is also obvious in the initial control effort, where both the sine and cosine terms are initially extremely distorted, changing faster or slower, depending on the heading direction.

6.2. Tracking by heading rate control, with disturbances

We demonstrate the tracking and stabilizing abilities of the controller by tracking a moving source with an open loop unstable system. Furthermore, in order to demonstrate the ability to control heading angle velocity, rather than the angle value directly, we implement the following scheme, in which an additional filter (76) of the function $J(x, y, t)$ has been introduced. The system is:

$$\dot{x} = x + 0.75y + \sqrt{\alpha\omega}\cos(\theta), \qquad x(0) = 1 \tag{73}$$

$$\dot{y} = 0.5x + 2y + \sqrt{\alpha\omega}\sin(\theta), \qquad y(0) = -1 \tag{74}$$

$$\dot{\theta} = \omega + k\omega^2(J - \eta), \qquad \theta(0) = 1.2 \tag{75}$$

$$\dot{\eta} = -\omega^2\eta + \omega^2 J, \quad \omega = 250 \tag{76}$$

$$r_x = \cos(t), \qquad r_y = \sin(2t) \tag{77}$$

$$J = (x - r_x)^2 + \left(y - r_y\right)^2, \qquad \alpha = 2, \qquad k = 10. \tag{78}$$

Intuitively, if one considers the combined θ, η dynamics as in (75), (76), then $\dot{\theta} = \omega + k\dot{\eta}$ and therefore $\theta(t) = \omega t + k\eta(t)$. Considering the transfer function $\eta = \frac{\omega^2}{s+\omega^2}J$, in the limit as ω approaches infinity, η approaches J, and so $\theta(t)$ approaches $\omega t + kJ$ as before. Note that the system is open loop unstable, with eigenvalues $\lambda_i = 2.3, 0.7$. Because of the disturbance and the non-zero velocity of $(r_x(t), r_y(t))$, we must use larger values of k, α, and ω. The simulation results are shown in Fig. 4.

7. Conclusions

This work recalls an important mathematical tool for the analysis of highly oscillatory systems, with which a new ES method's stability is derived. This new ES method provides a scheme in which the update rate of minimization or the control effort in the case of trajectory tracking/stabilization is bounded, despite uncertainty of the functions being minimized or the systems being controlled. Known bounds on update rates and control efforts are important for actual in-hardware control implementation. We demonstrate, without proof, a controller design in which the heading angle rate, rather than the angle directly, is controlled.

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